MATH 2230A Complex Variables with Applications

Solution and Remarks to Midterm 2.

1. (a) $\int_{\gamma_1} \frac{1}{z^2-1}\,dz = \int_{\gamma_1} \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)dz = \int_{\gamma_1} \frac{\frac{1}{2}}{z-1}\,dz + \int_{\gamma_1} \frac{-\frac{1}{2}}{z+1}\,dz$

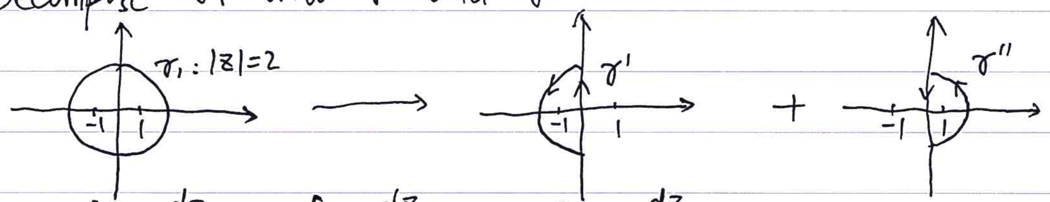
Noted that $\frac{1}{2}$ and $-\frac{1}{2}$ are analytic on $|z| \le 2$ and that 1 and -1 are interior to the circle $|z|=2$.

By C.I.F., we have $\int_{\gamma_1} \frac{\frac{1}{2}}{z-1}\,dz = 2\pi i \cdot \frac{1}{2} = \pi i$

$$\int_{\gamma_1} \frac{-\frac{1}{2}}{z+1}\,dz = 2\pi i \cdot \left(-\frac{1}{2}\right) = -\pi i$$

Thus, $\int_{\gamma_1} \frac{1}{z^2-1}\,dz = \pi i + (-\pi i) = 0$

(b) Decompose γ_1 into γ' and γ''



Then $\int_{\gamma_1} \frac{dz}{(z^2-1)^3} = \int_{\gamma'} \frac{dz}{(z^2-1)^3} + \int_{\gamma''} \frac{dz}{(z^2-1)^3}$

$$= \int_{\gamma_1} \frac{\frac{1}{(z-1)^3}}{(z+1)^3}\,dz + \int_{\gamma''} \frac{\frac{1}{(z+1)^3}}{(z-1)^3}\,dz$$

Noted that $\frac{1}{(z-1)^3}$ is analytic inside and on γ' and -1 is interior to γ'.

By C.I.F., we have $\int_{\gamma'} \frac{\frac{1}{(z-1)^3}}{(z+1)^3}\,dz = \frac{2\pi i}{2!}\left[\frac{1}{(z-1)^3}\right]''\Big|_{z=-1} = \frac{-12}{32}\pi i$

Similarly, $\int_{\gamma''} \frac{\frac{1}{(z+1)^3}}{(z-1)^3}\,dz = \frac{2\pi i}{2!}\left(\frac{1}{(z+1)^3}\right)''\Big|_{z=1} = \frac{12}{32}\pi i$

Thus, $\int_{\gamma_1} \frac{dz}{(z^2-1)^3} = 0.$

(c) On the circle γ_2, $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$

Thus, $dz = ie^{i\theta}\,d\theta$.

Then $\int_{\gamma_2} |z-1|\,|dz| = \int_0^{2\pi} |e^{i\theta}-1| \cdot |ie^{i\theta}|\,d\theta$

$$= \int_0^{2\pi} |\cos\theta + i\sin\theta - 1| \cdot |i\cos\theta - \sin\theta|\,d\theta$$

$$= \int_0^{2\pi} \sqrt{(\cos\theta-1)^2 + \sin^2\theta}\,d\theta$$

$$= \int_0^{2\pi} \sqrt{2-2\cos\theta}\,d\theta = \int_0^{2\pi} \sqrt{4\sin^2\tfrac{\theta}{2}}\,d\theta$$

$$= 2\int_0^{2\pi} \left|\sin\tfrac{\theta}{2}\right| d\theta$$

$$= 2\int_0^{2\pi} \sin\tfrac{\theta}{2}\,d\theta$$

$$= 2\left[-2\cos\tfrac{\theta}{2}\right]\Big|_0^{2\pi} = 8$$

(d) Noted that e^z is analytic on $|z|\le 1$ and 0 is interior to $|z|=1$

By C.I.F, we have

$$\int_{\gamma_2} \frac{e^z}{z^4}\,dz = \frac{2\pi i}{3!}\left(e^z\right)'''\Big|_{z=0} = \frac{\pi i}{3}$$

2. Solution: $(1+i)^{\frac{1}{3}} = e^{\frac{1}{3}\log(1+i)}$

$$= e^{\frac{1}{3}[\ln|1+i| + i\arg(1+i)]}$$

$$= e^{\frac{1}{3}[\ln\sqrt{2} + i(\frac{\pi}{4}+2k\pi)]}$$

$$= 2^{\frac{1}{6}} e^{\frac{i}{3}(\frac{\pi}{4}+2k\pi)} \qquad k\in\mathbb{Z}$$

When $k=0$, $w_0 = 2^{\frac{1}{6}} e^{\frac{\pi}{12}i}$

When $k=1$, $w_1 = 2^{\frac{1}{6}} e^{\frac{3}{4}\pi i}$

When $k=2$, $w_2 = 2^{\frac{1}{6}} e^{\frac{17}{12}\pi i}$

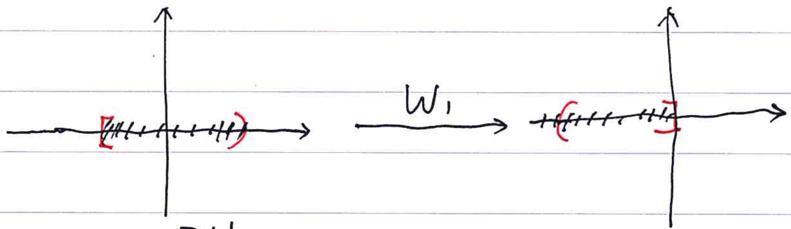
Therefore, the third roots of $1+i$ are w_0, w_1, w_2 find Δ

3. Solution:

Let $\frac{25}{9} = [\frac{1}{2}(\rho_0 + \rho_0^{-1})]^2$

Then $\rho_0 = \frac{1}{3}$ (since $\rho_0 < 1$) ☆ pay attention

Step 1:

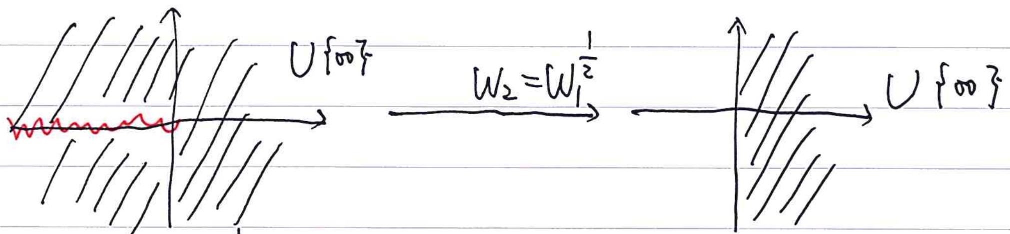


Let $w_1 = \frac{z+1}{z-1}$

Then $w_1([-1,1)) = (-\infty, 0]$

$w_1(\mathbb{C}\cup\{\infty\}\setminus[-1,1)) = \mathbb{C}\cup\{\infty\}\setminus(-\infty,0]$

Step 2:



Let $w_2 = w_1^{\frac{1}{2}}$

Then $w_2(\mathbb{C}\cup\{\infty\}\setminus(-\infty,0]) = \{x+iy \mid x>0\}\cup\{\infty\}$

(Method 1)

Step 3. Let $W_3 = \frac{W_2 - 1}{W_2 + 1}$

Then $W_3(\{x+iy \mid x>0\} \cup \{\infty\})$ = interior of $S_1 \cup \{\infty\}$

Step 4: Let $W_4 = \frac{1}{3} W_3$

Then W_4 maps interior of $S_1 \cup \{\infty\}$

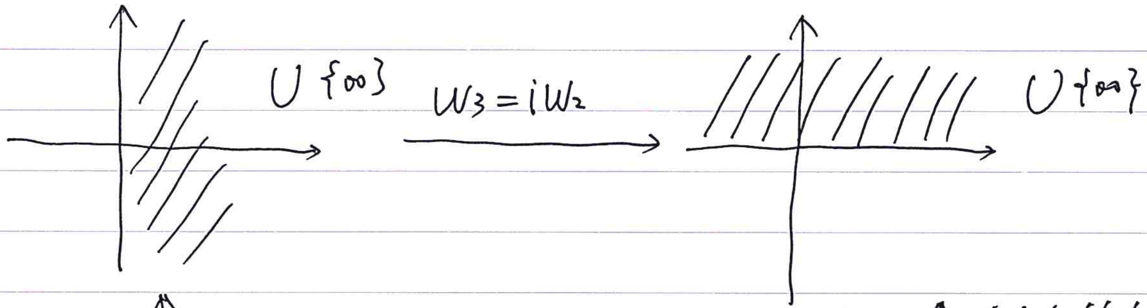
to $\{z \mid |z| < \frac{1}{3}\} \cup \{\infty\}$

Step 5: Let $W_5 = \frac{1}{2}(W_4 + 1/W_4)$

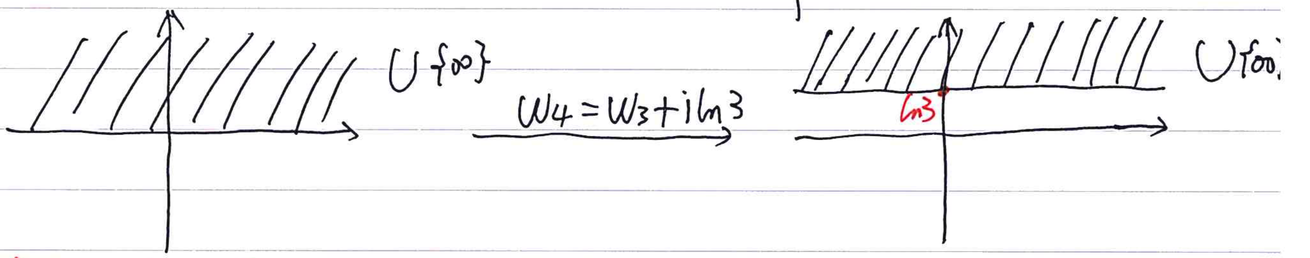
Then W_5 maps $\{z \mid |z| < \frac{1}{3}\} \cup \{\infty\}$ to exterior part of the ellipse $\frac{9x^2}{25} + \frac{9y^2}{16} = 1 \cup \{\infty\}$

(Method 2)

Step 3:



Step 4:



[why $y = \ln 3$?

Let $u = \sin x \cosh y$, $v = \sinh y \cos x$

Then $\frac{9u^2}{25} + \frac{9v^2}{25} = \frac{9}{25} \sin^2 x \cosh^2 y + \frac{9}{16} \sinh^2 y \cos^2 x$

$= \frac{9}{25} \sin^2 x \left(\frac{3+\frac{1}{3}}{2}\right)^2 + \frac{9}{16} \left(\frac{3-\frac{1}{3}}{2}\right)^2 \cos^2 x = 1$]

Step 5: